Avalanches and waves in the Abelian sandpile model

Maya Paczuski[1] and Stefan Boettcher[2]

[1]*Department of Physics, University of Houston, Houston, Texas 77204-5506*

[2]*Center for Nonlinear Studies, MS-B258, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*

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We numerically study avalanches in the two-dimensional Abelian sandpile model in terms of a sequence of waves of toppling events. Priezzhev *et al.* [Phys. Rev. Lett. **76**, 2093 (1996)] have recently proposed exact results for the critical exponents in this model based on the existence of a proposed scaling relation for the difference in sizes of subsequent waves, $\Delta s = s_k - s_{k+1}$, where the size of the previous wave s_k was considered to be almost always an upper bound for the size of the next wave s_{k+1}. Here we show that the significant contribution to Δs comes from waves that violate the bound; the average $\langle \Delta s(s_k) \rangle$ is actually negative and diverges with the system size, contradicting the proposed solution. [S1063-651X(97)50510-1]

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The sandpile was the first model introduced by Bak, Tang, and Wiesenfeld to demonstrate the principle of self-organized criticality [1]. Self-organized criticality describes a general property of slowly driven dissipative systems with many degrees of freedom to evolve toward a stationary state where activity takes place intermittently in terms of bursts spanning all scales up to the system size. The sandpile model has subsequently received a great deal of attention due in part to its potential for having a theoretical solution. Dhar showed that certain aspects of its behavior could be calculated exactly based on the Abelian symmetry of topplings. For instance, rigorous results have been obtained for the total number of allowed configurations on the attractor of recurrent states [2], for some height-height correlation functions [3,4], and later for the distribution of sizes of the last wave in an avalanche [5], among other quantities. Nevertheless, a formal solution for the all important critical exponents describing the distribution of avalanche sizes and durations in the Abelian sandpile model (ASM) has remained an elusive goal.

In a recent Letter [6] exact results were proposed for the distribution of avalanche sizes in the ASM, based on a decomposition of an avalanche into a sequence of ''waves'' of topplings. Specifically, the prediction was made that the asymptotic distribution of the number of topplings in an avalanche is $P(S) \sim S^{-\tau}$ with $\tau = 6/5$, and the distribution of the number of sites covered by an avalanche is $P(a) \sim a^{-\tau_a}$ with $\tau_a = 5/4$. The results of numerical simulations do not convincingly support these claims, although measuring the actual avalanche distribution exponents in this model is notoriously difficult.

Here we scrutinize the main assumption in the argument leading to the predicted exact results in Ref. [6] for the ASM. Based on careful numerical simulations we show that the fundamental assumption that the next wave usually is contained by the previous wave fails drastically. In particular, the exponent α as defined in [6] does not exist, and the difference in sizes of subsequent waves Δs is more often negative than positive. In fact, the negative contribution has a sufficiently fat tail that the average difference $\langle \Delta s(s_k) \rangle$ is negative and diverges with system size for all $s_k < s_{\rm co}$. The quantity $s_{\rm co}$ is the cutoff in waves sizes due to the finite size system, $s_{\rm co} \sim L^2$, where L is the linear extent of the system. In short, the physics is dominated by waves that are not contained within their predecessors. Thus the argument leading to the claimed exact results is incorrect.

The ASM consists of a square lattice of size L with a discrete number z_i of sand grains occupying each site. Initially, the lattice may be empty, and sand is dropped grain by grain at random sites. After each drop, all sites that exceed a critical threshold for stability, $z_i > z_c = 3$, are ''toppled'' by distributing a single grain of sand to each of their four nearest neighbors or, for boundary sites, over the edge of the lattice. Toppling proceeds for a number of time steps until all sites are stable again, and a new grain is dropped at a random site. Dropping sand represents an external driving force on the system whose impact is dissipated in intermittent sequences of toppling events, which are called avalanches. The number of toppling events following the addition of a single grain is the size S of an avalanche. Starting from an empty lattice, avalanches are initially rare and only of short duration. But the system fills up with sand to the point that many sites are close to threshold. Then, the system reaches a stationary state in which for any one grain dropped, on average, one grain must leave the system through the open boundaries. The grains are transported by avalanches that are now broadly distributed in both duration and extent over many orders of magnitude, only limited by the system's size. Thus, the system has self-organized into a critical (SOC) state with a highly correlated response to the external driving.

This model has a few other remarkable properties, most notably the fact that the order of toppling events during an avalanche is interchangeable (''Abelian'') without changing the final state of the system, which, for instance, allows an exact enumeration of the critical state [2]. Also, it was found that the domain spanned by a single avalanche is always compact, though with a fractal boundary [7]. Manna [8] introduced a different sandpile model, without Abelian symmetry, where the toppling grains are stochastically distributed to nearest neighbors so that the distribution is symmetric only on average. For some time it was believed, based on real space renormalization group arguments [9] and Manna's numerical simulation results, that the ASM was in the same universality class as the Manna model. But for both

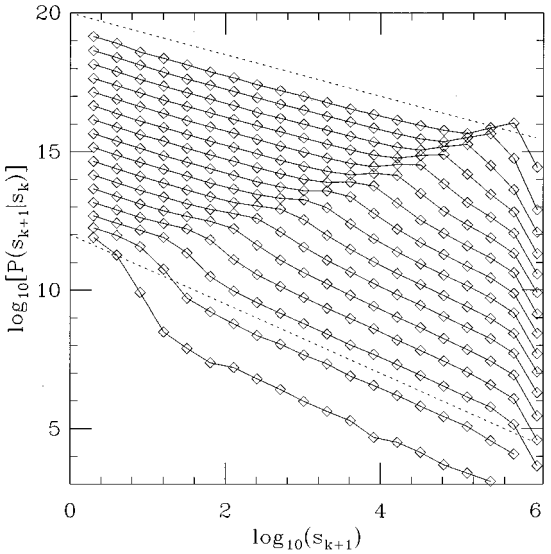

FIG. 1. Plot of the distribution $P(s_{k+1}|s_k)$ for the next wave to be of size s_{k+1}, given that the previous wave was of size s_k in a system of size $L=1024$. Each graph contains data for $2^m \leqslant s_k < 2^{m+1}$ for $m=4,5,\ldots,19$ from bottom to the top. To avoid overlaps, the graphs are offset. It appears that for each value of s_k, $P(s_{k+1}|s_k)$ initially falls like $s_{k+1}^{-3/4}$ (as the dashed line on top), but at a scale linear in s_k crosses over to a $s_{k+1}^{-5/4}$ falloff (as the lower dashed line).

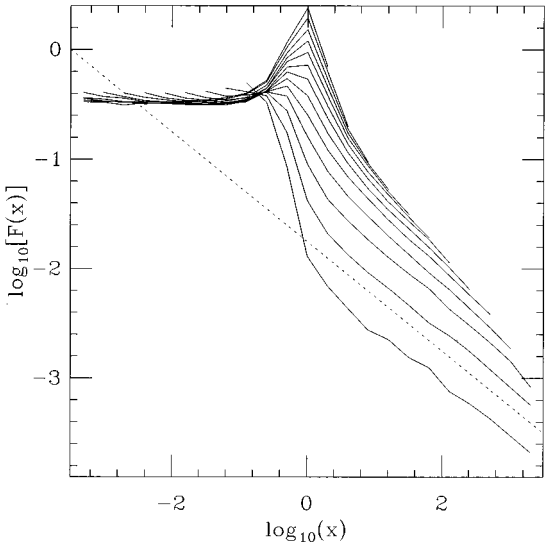

FIG. 2. Scaling collapse for the data in Fig. 1 according to Eq. (1), giving $F(x) \sim s_{k+1}^{3/4} P(s_{k+1}|s_k)$ as a function of $x=s_{k+1}/s_k$. The tail for each graph falls approximately like $x^{-1/2}$ (as the dashed line).

models the values of the distribution exponents, obtained by various extrapolations of the results from extensive numerical simulations, have barely converged to within 10% after many years of study. Recently, Ben-Hur and Biham [10] computed the geometrical scaling properties of avalanches instead of individual avalanche distribution exponents. Their results indicate that the ASM and the Manna model may belong to different universality classes. A survey of two-time autocorrelation functions in various SOC models [11] also reveals differences between both models: the ASM exhibits ''aging'' [12] while the Manna model does not [11,13].

Effort has focused recently on understanding avalanche dynamics in the ASM by decomposing the avalanche into a sequence of more elementary events. Dhar and Manna [5] introduced the notion of inverse avalanches, which were shown to be equivalent to a direct representation of avalanches in terms of waves of toppling events [14]. Ivashkevich, Ktitarev, and Priezzhev defined waves as follows: if the site (i) to which a grain was added becomes unstable, topple it once and then topple all other sites that become unstable, keeping the initial site (i) from toppling a second time. The set of sites that toppled thus far are called ''the first wave of topplings'' since every site can only topple once. After the first wave is completed the site (i) is allowed to topple the second time, not permitting it to topple again until the ''second wave of topplings'' is finished. The process continues until the site (i) becomes stable and the avalanche stops.

This elegant decomposition of avalanches into waves reveals many interesting features. First of all, the waves are individually compact, and each site that topples in a wave topples exactly once in that wave. As a result, the state of the system after a wave is exactly the same as the state before the wave except at sites on the single closed boundary of the wave, which separates sites that toppled in that wave from sites that did not. Just inside the wave boundary a trough relative to the previous heights appears, whereas just outside the boundary a hill appears where sand was transport outside of the wave. Thus the sequence of topplings in the next wave will follow exactly the sequence of topplings in the previous wave until the wave first reaches the prior wave's boundary, at which point the sequence may differ. Priezzhev, Ktitarev, and Ivashkevich [6] argued that generally subsequent waves are spatially contained within the previous waves because of the trough at the boundary. Their analysis only considers waves that are contained within previous waves and they ''neglect the overlapping of waves and deal only with the decrease of wave size'' [6]. Using spanning tree arguments together with this assumption, they find that the size difference between subsequent waves $\langle \Delta s \rangle = s_k - s_{k+1}$ is positive, finite in the limit of large system size, and obeys a scaling relation $\Delta s \sim s_k^{\alpha}$. Key to the argument leading to Eq. (12) in Ref. [6] is the length of the boundary Γ of the previous wave s_k, which is presumed to contain the subsequent wave s_{k+1}. Here we show that the main contribution to Δs comes from waves that escape the boundary of their preceding wave, and are bounded only by the system size. In fact, the average $\langle \Delta s(s_k) \rangle$ is negative and diverges to infinity as the system size increases.

We have simulated about 10^7 waves in L^2 systems up to $L=1024$. To ensure the accuracy of our numerical simulations, we have reproduced a variety of previously obtained exact results for the distribution of waves. For instance, our data yield the $s^{-11/8}$ power law that was derived by Dhar and Manna [5] for the distribution of the very last wave in each avalanche. We also found the $1/s$ behavior for the distribution of all waves [14].

To determine Δs we recorded the distribution of subsequent waves for a given size of the preceding wave $P(s_{k+1}|s_k)$ as shown in Fig. 1. In Fig. 2 we show the data collapse where the horizontal axis is now $x=s_{k+1}/s_k$. There

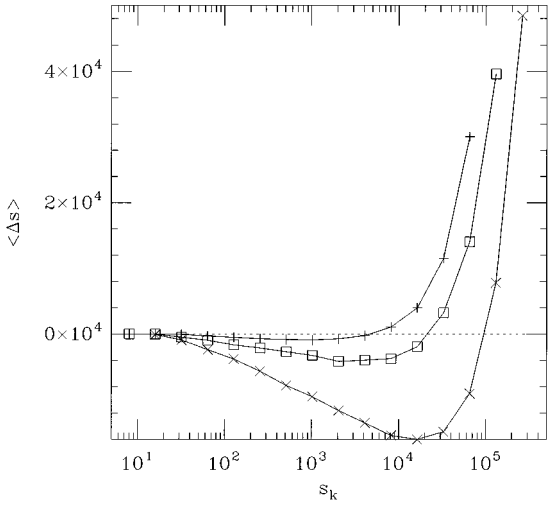

FIG. 3. Plot of $\langle \Delta s \rangle = s_k - s_{k+1}$ as a function of the previous wave s_k, obtained from $P(s_{k+1}|s_k)$ (see Fig. 1) via Eq. (2) for system size $L = 2^8$, 2^9, and 2^{10} from top to bottom. Initially, $\langle \Delta s \rangle$ is negative and falling for increasing s. Closer to the cutoff s_{co} the linearly rising term in Eq. (3) dominates. In each case, the graph turns positive at $s_k \approx L^2/10$.

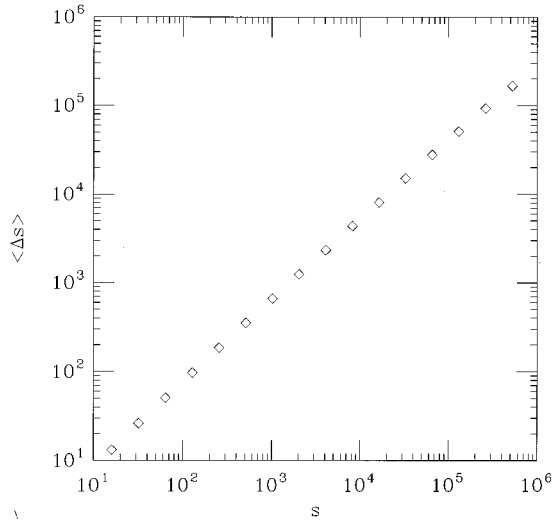

FIG. 4. Plot of $\langle \Delta s \rangle$ as a function of the previous wave s, but leaving out all data where Δs is negative. We chose $L = 512$ to compare with Fig. 1 in Ref. [6].

are two regimes separated by a turning point near one. There is clear evidence of a fat power law tail for $x \gg 1$, which will dominate the average Δs for each value of s_k. The data are sufficiently well represented by a scaling form

$$P(s_{k+1}|s_k) \sim s_{k+1}^{-\beta} F\left(\frac{s_{k+1}}{s_k}\right), \tag{1}$$

where $F(x \to 0) \to 1$ and $F(x \gg 1) \sim x^{-r}$. The data collapse indicates that $\beta \simeq 3/4$ and $r \simeq 1/2$.

From Eq. (1), the computation of $\langle \Delta s(s_k) \rangle = \langle s_k - s_{k+1} \rangle$ leads immediately to

$$\langle \Delta s(s_k) \rangle = s_k - s_k \int^{s_{co}/s_k} dx x^{1-\beta} F(x), \tag{2}$$

where s_{co} is the cutoff in wave sizes from the finite system size; $s_{co} \sim L^2$. For $s_k \ll s_{co}$ this gives $\langle \Delta s(s_k) \rangle = s_k - C s_{co}^{2-\beta-r} s_k^{\beta+r-1}$ or, using our values for β and r,

$$\langle \Delta s(s_k) \rangle = s_k - C s_{co}^{3/4} s_k^{1/4}, \tag{3}$$

where C is a positive number that depends on the details of the function F. *It is important to note here that $\langle \Delta s \rangle$ is negative for all s_k up to a scale L^2, and that it diverges with the system size.* Our numerical measurement for $\langle \Delta s \rangle$ shown in Fig. 3 confirms the above analysis.

If we (erroneously) exclude from the data set all waves that are larger than their predecessor, i.e., eliminating those waves that contribute to a negative Δs, then we reproduce (in Fig. 4) the plot given in Fig. 1 of Ref. [6].

Furthermore, a more detailed analysis of subsequent waves that *are* smaller, in terms of number of topplings s, than their predecessor reveals that in most cases the following (smaller) wave still, more often than not, exceeds the confines of the (larger) previous wave at some points on the boundary. The smaller waves actually do escape the boundary of their predecessor. As the system size grows, the fraction of consecutive waves that violate the assumption of Ref. [6] also grows.

In summary, we have shown that the analysis in Ref. [6] is fundamentally flawed because it deals only with waves of decreasing size, whereas the dominant contribution to Δs comes from the ''fat tails'' in the distribution $P(s_{k+1}|s_k)$ where waves escape the boundary of their predecessor explicitly. Yet, our numerical data seems to indicate that a certain degree of regularity in the distribution of consecutive waves exists, leading to what appears to be exact values for $\beta = 3/4$ and $r = 1/2$. It may be possible that the spanning tree arguments used in Ref. [6] can be generalized to include the dominant contribution from overlapping waves. The appearance of the apparently simple exponents β and r gives us some hope that an exact solution using the elegant decomposition of avalanches into waves can be discovered.

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